12th Oral Exam of S.-T. Yau College Student Mathematics Contests 2021

Analysis and differential equations
Individual Contest

(∗ The forth and fifth problems are optional)

1. Let $\mathbb{Z}$ be the set of integers. Recall that $\vec{a} := \{a_j\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z})$ if and only if $(\sum_{j\in\mathbb{Z}} |a_j|^q)^{\frac{1}{q}} < \infty$. Answer the following questions and justify your answers:

   (i) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ continuous ?

   (ii) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ compact?

   (iii) Is the embedding $l^2(\mathbb{Z}) \to l^4(\mathbb{Z})$ compact modulo translation? More precisely, let $\{\vec{a}_n\}_n$ be a sequence in $l^2(\mathbb{Z})$ with $\vec{a_n} = \{a_{j,n}\}_{j\in\mathbb{Z}}$. Can one find $k_n \in \mathbb{Z}$ for $n \geq 1$, such that $\{\vec{b}_n\}_n$, with $\vec{b}_n := \{b_{j,n}\}_{j\in\mathbb{Z}}$ and $b_{j,n} = a_{j-k_n,n}$ has a convergent subsequence in $l^4(\mathbb{Z})$?

2. Let S^1 be the unit circle in $\mathbb{R}^2$. For any $v \in S^1$, let $\pi_v : \mathbb{R}^2 \to \mathbb{R} : \pi_v(x) = \langle v, x\rangle$. Let A, B be two bounded open convex sets in $\mathbb{R}^2$ and λ_1 be the Lebesgue measure on $\mathbb{R}^1$.

   (i) If for all $v \in S^1$, $\lambda_1(\pi_v(A)) = \lambda_1(\pi_v(B))$, can one conclude that $A = B$? Can one conclude that $A = B$ modulo isometries?

   (ii) If for all $v \in S^1$, $\pi_v(A) = \pi_v(B)$, can one say that $A = B$?

   Justify your answers.

3. Assume that $\rho \in C_0^1(\mathbb{R}^3)$ satisfies $\rho(x) \geq 0$ for $x \in \mathbb{R}^3$ and

   $$\nabla(\rho^{4/3}) + \rho\nabla\phi = 0, \ \ on\ \mathbb{R}^3$$

   where

   $$\phi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}dy.$$

   Evaluate the integral

   $$\int_{\mathbb{R}^3} (3\rho^{4/3} + \frac{1}{2}\rho\phi)dx,$$

   and justify your answer.

4*. Let $D = \{z, |z| < 1\}$. Determine $Aut(D)$, the group of holomorphic automorphisms of the unit disk.

5*. Assume that H is space with inner product, and $x_k \xrightarrow{w} x$ $(k \to \infty)$. Prove that there exists a subsequence $\{x_{k_n}\} \subset \{x_k\}$ such that $\frac{x_{k_1}+x_{k_2}+\cdots+x_{k_n}}{n} \to x (n \to \infty)$.